Stability Theorems for Stochastic Differential Equations (S.D.E.'s) with Memory (Part 2)

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Abstract

Here stochastic differential equations with memory means delay stochastic differential equations. In this work we have proved a stability theorem (for the diffusion of the S.D.E. (in [3]) which is an extension of the stability theorem of section three (in [3])to a stability theorem in history space. The work in this section is done by suitable modifications of the corresponding work in [12]. In [3] we have formulated an example of the main delay stochastic differential equation , see [3] and [1]. The example which we have considered is of the following form:

$$d\begin{pmatrix} x_1(t) \\ x_2(t) \\ \tilde{x}^t \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1(t) + \int_0^\infty e^{-s}(x_1^t(s) - x_1(t))ds \\ -(\tilde{x}^t)' - x_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \alpha\|\tilde{x}^t\| \\ 0 \end{pmatrix} dW_t$$

where the ordered triple $\left(x_1(t), x_2(t), \widetilde{x}^t\right)$ can be considered as representing position, velocity and history of position respectively. We will call the space containing this triple "the history space X". In section three of [3]) we have proved a stability theorem for a diffusion of a S.D.E. in $\mathbf{R}^n$. With a suitable choice of Lyapunov functional we have proved that the motion will finally come to rest at the origin. In section four of [3]),we have extended the space $\mathbf{R}^n$ of section three in [3]) to a history space, i.e. to a space with three components; position, velocity and history of position. Also in [3])we have formulated our delay S.D.E. on this history space X and also we found the generator of the diffusion. The present work was suggested by Prof. Maassen,J.D.M.,Katholik University of Nijmegen,The Netherlands.

1 Introduction

By "with memory" we mean a delay S.D.E. in which the initial process is defined on an interval of time in the past and not at a particular point as in the ordinary S.D.E.'s. Stochastic Differential Equations with memory serve as models of noisy physical processes whose time evolution depends on their past history. In physics, lazer dynamics with delayed feed back is often invistigated as well as the dynamics of noisy bistable systems with delay. In biophysics,stochastic equations are used to model delayed visual feed back systems or human postural way. For more details see the website of Prof. Salah-E.A.Mohammed namely "sfde.math.siu.edu". By "with memory" we mean a S.D.E. in which the initial process is defined on an interval of time in the past and not at a particular point as in the ordinary S.D.E.'s.

2 The stability theorem in history space

Here we shall prove an extension of the stability theorem 1 (in [3]) to a stability theorem in the history space X defined in section four of [3].

Let $Q_1 = \{x : V(x) < q\} = \{x : \|x\| < \sqrt{2q}\}$. Then

$$\begin{aligned} X_t \in Q &\Longrightarrow V(X_t) < q \\ &\Longleftrightarrow \tfrac{1}{2} \|X_t\|^2 < q \\ &\Longleftrightarrow \|X_t\| < \sqrt{2q}. \end{aligned} \tag{2.1}$$

It is clear that the set Q is bounded.

Before stating Theorem 1 we need the following settings (see [12], Theorem 4.2). In general the set Q_1 defined in (2.1) is not bounded while we have defined the generator on bounded sets only. However we can deal with the situation (appearing in our application) in which $V(X_t) \leq q$ implies $\|X_t\| \leq M$ where M is a constant independent of the initial history X_0. Then if we consider only the initial histories belonging to a bounded set in Q_1, there exist a bounded set $U \subset X$ such that $X_t \in U$ for all $t \leq \tau_1$ (τ_1 is the exit time from Q_1). More precisely, we let V, Q_1 and τ_1 be as above and let X_t be the solution of the S.D.E. (iv.5) (in [3]) with coefficients a and b satisfying the inequalities (iv.6) and (iv.7) (in [3]). We assume that for any bounded set D in X there is a constant $K_3 > 0$ such that for $x = X_0$ in $D \bigcap Q_1 = Q_D$, the norm of X_t is bounded by K_3 for $0 \leq t \leq \tau_1$:

$$\|X_t\| \leq K_3 \quad \text{for} \quad 0 \leq t \leq \tau_1 \tag{2.2}$$

Put $B = \{x \in X : \|x\| \leq K_3\}$ and $Q := Q_1 \bigcap B \supset Q_D$. Let τ be the exit time from Q and let τ_B be the exit time from B ($\tau_B \geq \tau$) and let $\bar{b}$ and $\bar{\sigma}$ be such that $\bar{b}(x) = b(x)$, $\forall x \in B$ and $\bar{\sigma}(x) = \sigma(x)$ $\forall x \in B$ and such that $\bar{b} \equiv 0$, $\quad \bar{\sigma} \equiv 0$ outside a neighborhood U of B. The process $\bar{X}_t$ of the S.D.E. (iv.5)(in [3]) with coefficients $\bar{b}$, $\bar{\sigma}$ is regarded as having phase space Y as constructed in lemma (3.B) 0f [12]. Also by Theorem 2.3 of [12], if $x = X_0 \in B$, then $\bar{X}_t = X_t$ for $t \leq \tau_B$. Let $\bar{A}$ be the weak infinitesimal operator of $\bar{X}_t$. Let X_t denote the process $\bar{X}_{\tau \wedge t}$ $(= X_{\tau \wedge t})$ and let A be its infinitesimal generator. Also for a fixed $x = X_0 \in Q$ let $\Omega_Q = \Omega_{Q,x}$ denote the set

$$\Omega_Q = \left\{ \omega \in \Omega : \sup_{t \geq 0} V(X_t(\omega)) < q \right\} \tag{2.3}$$

Let τ be the exit time from Q, i.e. $\tau = \inf\{t : X_t \notin Q\}$. Also let $\tau(t) = \tau \wedge t$.

1 Theorem (The Stability Theorem). *Under the conditions described above, and with A and V as in section four in [3], the following result is valid. If*

(i) $AV(x) \leq 0$ *for all* $x \in Q$ *and*

(ii) AV *is uniformly continuous on* Q, *then*

(a) $\mathbf{P}(\Omega_Q) \geq 1 - \frac{V(x)}{q} \qquad \forall x \in Q$

(b) $AV(X_t) \to 0$ *as* $t \to \infty$ *a.s. in* Ω_Q

Since, by (iv.14)(in [3]), $AV(X_t) = \frac{1}{2}\left(\alpha^2 - 1\right)\|\tilde{x}^t\|^2$, from (b) it follows that $\lim_{t\to\infty}\left\|\tilde{x}^t\right\|^2 = 0$.

We shall first check that conditions (i) and (ii) are satisfied with V and A as in equations (iv.8) and (iv.14) (in [3]) respectively. By (iv.14) (in [3]) it is easy to see that

$$AV(x) \leq 0 \quad \forall x \in Q \quad \text{as } \alpha^2 < 1 \text{ and } \|\tilde{x}\|^2 \geq 0 \tag{2.4}$$

Thus condition (i) is satisfied.

To check condition (ii), we shall only check that AV is Lipschitz, because then it will be uniformly continuous, i.e. we need only to check that

$$|AV(x) - AV(y)| \leq K\|x-y\|_X \qquad \forall x,\ y \in Q.$$

Now

$$\begin{aligned}
|AV(x) - AV(y)| &= \left|\tfrac{1}{2}\left(\alpha^2-1\right)\|\tilde{x}\|^2 - \tfrac{1}{2}(\alpha^2-1)\|\tilde{y}\|^2\right| \quad \text{(by (iv.14) in [3])}\\
&= \left|\tfrac{1}{2}\left(\alpha^2-1\right)\right|\left|\|\tilde{x}\|^2 - \|\tilde{y}\|^2\right|\\
&= K'\left|\|\tilde{x}\|^2 - \|\tilde{y}\|^2\right|\\
&\text{where } K' = \tfrac{1}{2}(1-\alpha^2) \text{ is a positive constant}\\
&= K'\left|\left(\|\tilde{x}\| - \|\tilde{y}\|\right)\left(\|\tilde{x}\| + \|\tilde{y}\|\right)\right|\\
&\leq K'\left|\left(\|\tilde{x}\| - \|\tilde{y}\|\right)\left(\|x\| + \|y\|\right)\right|\\
&\leq K'2\sqrt{2q}\,\left|\|\tilde{x}\| - \|\tilde{y}\|\right| \quad \text{as } \|x\| < \sqrt{2q} \text{ and } \|y\| < \sqrt{2q}\\
&= K\left|\|\tilde{x}\| - \|\tilde{y}\|\right|\\
&\ \text{where } K = 2K'\sqrt{2q} \text{ is a real constant}\\
&\leq K\|\tilde{x} - \tilde{y}\|\\
&\leq K\left(\|\tilde{x}-\tilde{y}\|^2 + |x_1 - y_1|^2 + |x_2 - y_2|^2\right)^{\frac{1}{2}}\\
&= K\|x-y\|_X .
\end{aligned}$$

Thus AV is Lipschitz and hence uniformly continuous on Q.

Proof of Theorem 1. Now it can be easily checked that the assumptions of Theorem 1(in [3]) where $X_t \in \mathbf{R}^n$ are implied by the assumptions of Theorem 1, where $X_t \in$ history space X. Hence using a similar argument as in the proof of Theorem 2.1 we find that $V(X_t)$ is a supermartingale by using Dynkin's formula and the equations (iv.12) and (iv.13) (in [3]) . Thus for $x \in Q$ we have by condition (i) of this theorem.

$$\mathbf{E}_x V(X^x_{\tau(t)}) - V(x) = \mathbf{E}_x \int_0^{\tau(t)} AV(X_s)ds \leq 0.$$

Hence $\mathbf{E}_x V(X_t) \leq V(x)$. Also by the martingale inequality we have

$$\begin{aligned}
\mathbf{P}_x(\Omega_Q) &= \mathbf{P}_x\{\omega : \sup_{t\geq 0} V(X_t) < q\}\\
&= 1 - \mathbf{P}_x\{\omega : \sup_{t\geq 0} V(X_t) \geq q\}\\
&\geq 1 - \tfrac{V(x)}{q}.
\end{aligned}$$

Also by the martingale convergence theorem we find that $V(X_t)$ converges to a non-negative random variable with probability 1. Thus $\|X_t\|^2 \to v$ with probability 1.

To prove part (b) of this theorem we shall use a method similar to that used in the proof of Theorem (4.2) of Mizel and Trutzer ([12]). Now define a map $k : X \to R^+$ by $k(x) = \frac{1}{2}(1-\alpha^2)\|\tilde{x}\|^2$. Hence $AV(X_t) = -k(X_t)$. Now it is easy to see that $k(x)$ is Lipschitz as AV is Lipschitz on Q. Also it can easily be seen that $k(x)$ is Lipschitz on the set $R_\delta = \{x : k(x) < \delta\}$ where δ is a positive real number, and hence k(x) is uniformly continuous on the set R_δ. Hence all the assumptions of Lemma (4.A) of Mizel and Trutzer [12] are satisfied and hence all its conclusions also hold for $X_t \in X$. Similarly if we prove that the condition :

$$P_x\left\{\tau \geq t+h, \sup_{0\leq s\leq h} \left\|\bar{X}_t - \bar{X}_{t+s}\right\| > \epsilon\right\} \to 0 \quad \text{as} \quad h \to 0 \tag{2.5}$$

uniformly in t (sufficiently large) holds for each $X_0 = x \in Q_D$, then all the hypotheses of Theorem 1 of Mizel and Trutzer [12] are satisfied and hence it's conclusion also holds for our X_t namely $k(X_t) \to 0$ as $t \to \infty$ a.s in Ω_Q. To prove (2.5) we have

$$P_x\left\{\tau \geq t+h, \sup_{0\leq s\leq h} \left\|\bar{X}_t - \bar{X}_{t+s}\right\| > \epsilon\right\} \leq \frac{1}{\epsilon^2}\mathbf{E}\left\{\sup_{0\leq s\leq h} \|X_t - X_{t+s}\|^2 \,|\tau \geq t+h\right\} \tag{2.6}$$

Note that we will omit the bar in $\bar{X}_t$ and write X_t for $\bar{X}_t$. Now for $\tau \geq t+h$ and $s \leq h$,we have by using equation(iv.5) (in [3]):

$$\begin{aligned}
\|X_t - X_{t+s}\|^2 &= \left\|-\int_t^{t+s} b(X_u)du - \int_t^{t+s} \sigma(X_u)dW_u\right\|^2 \\
&\leq 2\left|\int_t^{t+s} b(X_u)du\right|^2 + 2\left|\int_t^{t+s} \sigma(X_u)dW_u\right|^2.
\end{aligned} \tag{2.7}$$

Now by Hölder inequality, the inequality (2.2) of Mizel and Trutzer [12], and inequality (iv.6)(in [3]) and inequalities(2.2) and (2.7) we have

$$\begin{aligned}
& \mathbf{E}\left\{\sup_{0\leq s\leq h} \|X_t - X_{t+s}\|^2 \quad |\tau \geq t+h\right\} \\
\leq\ & 2\mathbf{E}\left\{\sup_{0\leq s\leq h} s\int_t^{t+s} |b(X_u)|^2 du \quad |\tau \geq t+h\right\} \\
& +2\mathbf{E}\left\{\int_t^{t+h} |\sigma(X_u)|^2 du \quad |\tau \geq t+s\right\} \\
\leq\ & 2h\int_t^{t+h} K_1^2(\|X_u\|+1)^2 du + 2\int_t^{t+h} K_1^2(\|X_u\|+1)^2 du \\
\leq\ & 2h\int_t^{t+h} K_1^2(K_3+1)^2 du + 2\int_t^{t+h} K_1^2(K_3+1)^2 du \\
=\ & 2K_1^2(K_3+1)^2h^2 + 2K_1^2(K_3+1)^2h \\
=\ & 2K_1^2(K_3+1)^2(h^2+h) \\
=\ & \mathcal{O}(h) \quad \text{as} \quad h \to 0 \quad \text{uniformly in t.}
\end{aligned} \tag{2.8}$$

Thus by (2.6) and (2.8) we obtain that

$$P_x\left\{\tau \geq t+h, \sup_{0\leq s\leq h} \left\|\bar{X}_t - \bar{X}_{t+s}\right\| > \epsilon\right\} \to 0 \quad \text{as} \quad h \to 0, \tag{2.9}$$

uniformly in t (sufficiently large). Hence we have $k(X_t) \to 0$ as $t \to \infty$ a.s. Thus $AV(X_t) = -k(X_t) = \frac{1}{2}(\alpha^2-1)\|\tilde{x}^t\|^2 \to 0$ as $t \to \infty$ a.s. on Ω_Q. Hence we have $\|\tilde{x}^t\|^2 \to 0$ as $t \to \infty$ a.s. on Ω_Q. Thus Theorem 1 is proved.

Now by Dynkin's formula we have

$$\mathbf{E}V(X_t^x) = \mathbf{E}V(x) + \mathbf{E}\int_0^t AV(X_s^x)ds$$

Thus by equation(iv.12)(in [3]) we have

$$\frac{1}{2}\mathbf{E}\left\|X_t\right\|^2 = \frac{1}{2}\mathbf{E}\left\|x\right\|^2 + \frac{1}{2}(\alpha^2-1)\int_0^t \mathbf{E}\left\|\tilde{x}^u\right\|^2 du \tag{2.10}$$

and hence

$$\frac{d}{dt}\mathbf{E}\left\|X_t\right\|^2 = -K\mathbf{E}\left\|\tilde{x}^t\right\|^2 < 0 \tag{2.11}$$

where $K = 1-\alpha^2$ is a positive constant. Hence $\mathbf{E}\left\|X_t\right\|^2$ is decreasing $\forall t \geq 0$. Thus $\mathbf{E}\left\|X_t\right\|^2 \to \mathbf{E}v$ where $v \geq 0$.

Since $\mathbf{E}\left\|\tilde{x}^u\right\| \leq \mathbf{E}\left\|X_u\right\|$, then by the differential equation (2.11) it follows that

$$\frac{d}{dt}\mathbf{E}\left\|X_t\right\|^2 + K\mathbf{E}\left\|X_t\right\|^2 \geq 0 \tag{2.12}$$

Now by (2.12) it follows that the solution process X_t satisfies

$$\mathbf{E}\left\|X_t\right\|^2 \geq C\exp\left(-Kt\right)$$

Now at $t=0$; $\quad \mathbf{E}\left\|X_t\right\|^2 = \mathbf{E}\|x\|^2$ and hence $C = \mathbf{E}\|x\|^2$
Thus

$$\mathbf{E}\left\|X_t\right\|^2 \geq \mathbf{E}\|x\|^2 \exp\left(-Kt\right) \tag{2.13}$$

Hence, it follows that $\mathbf{E}\left\|X_t\right\|^2$ takes values between zero and $\mathbf{E}\|x\|^2$. Now by (2.11), (2.12) and (2.13) it follows that $\mathbf{E}\left\|X_t\right\|^2$ is decreasing and it starts at $t=0$ with the value $\mathbf{E}\|x\|^2$ and decreases till it reaches some value larger than or equal to zero. $\square$

2.1 More on stability

In Theorem 1 we have proved that $\left\|\tilde{x}^t\right\|^2 \to 0$ as $t \to \infty \quad$ a.s. on Ω_Q. Our final aim is to see the behavior of $\|X_t\|^2$ $(= x_1(t)^2 + x_2(t)^2 + \left\|\tilde{x}^t\right\|^2)$ as $t \to \infty$. Now it remains to study $\lim_{t\to\infty}(x_1(t)^2 + x_2(t)^2)$. We shall first state and prove the following lemma.

2 Lemma. *Put* $A(t) = A(0) - \int_0^t \alpha\left\|\tilde{x}^u\right\| \sin u\, dW_u - \int_0^\infty e^{-s}\int_0^t \tilde{x}^u(s)\sin u\, du\, ds$, *and* $B(t) = B(0) + \int_0^t \alpha\left\|\tilde{x}^u\right\| \cos u\, dW_u + \int_0^\infty e^{-s}\int_0^t \tilde{x}^u(s)\cos u\, du\, ds$. *Then*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \tag{2.14}$$

Moreover, $x_1(0) = A(0)$, *and* $x_2(0) = B(0)$. *Consequently* $\forall t \geq 0$

$$x_1(t)^2 + x_2(t)^2 = A(t)^2 + B(t)^2 \tag{2.15}$$

and hence

$$\sup_{t\geq 0}\left(x_1(t)^2 + x_2(t)^2\right) < \infty \quad \text{if and only if } \sup_{t\geq 0}\left(A(t)^2 + B(t)^2\right) < \infty. \tag{2.16}$$

In addition,

$$\mathbf{E}\int_0^\infty \left\|\tilde{x}^t\right\|^2 dt < \infty \tag{2.17}$$

and hence it follows that

$$\sup_{t>0} \mathbf{E} \left| \int_0^\infty e^{-s} \int_0^t \tilde{x}^u(s) e^{iu} du\, ds \right|^2 < \infty \qquad \textit{and} \tag{2.18}$$

$$\sup_{t>0} \mathbf{E} \left(\int_0^\infty e^{-s} \int_0^t \tilde{x}^u(s) \cos u\, du\, ds 1_{\Omega_Q} \right)^2 \qquad \textit{and} \tag{2.19}$$

$$\sup_{t>0} \mathbf{E} \left(\int_0^\infty e^{-s} \int_0^t \tilde{x}^u(s) \sin u\, du\, ds \right)^2 \tag{2.20}$$

are finite as well. Of course, here $i = \sqrt{-1}$.

It probably follows that the limits $\lim_{t\to\infty} A(t)$ and $\lim_{t\to\infty} B(t)$ exist $\mathbf{P}$-almost surely.

Proof. Put

$$\begin{aligned} &(x_1 + ix_2)(t) \\ &= e^{-it}(x_1 + ix_2)(0) + ie^{-it} \left(\int_0^\infty e^{-s} \int_0^t \tilde{x}^u(s) e^{iu} du\, ds + \int_0^t \alpha \|\tilde{x}^u\| e^{iu} dW_u \right). \end{aligned} \tag{2.21}$$

Hence,

$$d(x_1 + ix_2)(t) = -i(x_1 + ix_2)(t)dt + i \int_0^\infty e^{-s} \tilde{x}^u(s) ds\, dt + i\alpha \|\tilde{x}^t\| dW_t. \tag{2.22}$$

Thus (2.21) satisfies the delay S.D.E. (iv.2) (in [3])
Now put

$$z(t) = (x_1 + ix_2)(t), \quad F(t) = (A + iB)(t). \tag{2.23}$$

Then by (2.21) it follows that

$$z(t) = e^{-it}(A + iB)(t) = -ie^{-it}(B - iA)(t). \tag{2.24}$$

Hence (2.14) follows. Moreover it follows that $x_1(0) = A(0)$ and $x_2(0) = B(0)$. Now by (2.24) we find that equation (2.15) holds and hence (2.16) also holds. Now we shall prove inequalities (2.17) and (2.18). Now by using the fact that $\|x\|^2 = x_1(0)^2 + x_2(0)^2 + \|\tilde{x}^0\|^2$ and equations (2.10) and (2.15) we have $\forall t \geq 0$

$$\begin{aligned} (1 - \alpha^2)\mathbf{E} \int_0^t \|\tilde{x}^u\|^2 du &\leq \mathbf{E} \left\{ A(t)^2 + B(t)^2 + \|\tilde{x}^t\|^2 \right\} + (1 - \alpha^2)\mathbf{E} \int_0^t \|\tilde{x}^u\|^2 du \\ &= \mathbf{E} \left\{ A(0)^2 + B(0)^2 + \|\tilde{x}^0\|^2 \right\} \\ &< \infty \end{aligned} \tag{2.25}$$

Hence inequality (2.17) follows. Now we have

$$(B - iA)(t) = (B - iA)(0) + \int_0^t \alpha \|\tilde{x}^u\|\, e^{iu} dW_u + S(t) \tag{2.26}$$

where $S(t) = \int_0^t \int_0^\infty e^{-s} \tilde{x}^u(s) ds e^{iu} du$.

Equation (2.26) can be written as

$$S(t) = B(t) - iA(t) - B(0) + iA(0) - \int_0^t \alpha \|\tilde{x}^u\|\, e^{iu} dW_u \tag{2.27}$$

Then by equations (2.10), (2.17) and (2.27) and using the properties of the Ito integral we get

$$\begin{aligned}
\sqrt{\mathbf{E}|S(t)|^2} &\leq \sqrt{\mathbf{E}(B(t)^2+A(t)^2)}+\sqrt{\mathbf{E}(B(0)^2+A(0)^2)} \\
&\quad +\sqrt{\mathbf{E}\left|\int_0^t \alpha\|\tilde{x}^u\|\, e^{iu} dW_u\right|^2} \\
&\leq 2\sqrt{\mathbf{E}(B(0)^2+A(0)^2)}+\sqrt{\mathbf{E}\|\tilde{x}^0\|^2}+\alpha\sqrt{\mathbf{E}\int_0^t \|\tilde{x}^u\|^2 du} \\
&\leq 2\sqrt{\mathbf{E}(B(0)^2+A(0)^2)}+\sqrt{\mathbf{E}\|\tilde{x}^0\|^2}+\alpha\sqrt{\mathbf{E}\int_0^\infty \|\tilde{x}^u\|^2 du} \\
&< \infty
\end{aligned} \tag{2.28}$$

Hence (2.18) follows, and hence also (2.19) and (2.20). To study $\lim_{t\to\infty}(x_1(t)^2+x_2(t)^2)$ rewrite equation (2.21) as follows:

$$\begin{aligned}
e^{it}z(t)-z(0) &= i\int_0^\infty e^{-s}\int_0^t \tilde{x}^u(s)e^{iu}du\, ds + i\int_0^t \alpha\|\tilde{x}^u\|^2 e^{iu}\, dW_u \\
&= iS(t)+iT(t)
\end{aligned} \tag{2.29}$$

where $S(t)=\int_0^\infty e^{-s}\int_0^t \tilde{x}^u(s)e^{iu}du\, ds$ and $T(t)=\int_0^t \alpha\|\tilde{x}^u\|e^{iu}\, dW_u$.
Put

$$Q(t)=iS(t)+iT(t) \tag{2.30}$$

Now we shall show that $Q(t)$ is a Dirichlet process, by checking that $Q(t)$ satisfies the conditions of Definition 1 in [4]. By letting $\pi(0,t)=(0=t_0<t_1<...<t_{N+1}=t)$ be a finite partition of the time interval $[0,t]$ and letting $\delta=\sup_k|t_{k+1}-t_k|$, we have,

$$\begin{aligned}
\sum_{k=0}^N \mathbf{E}\left|S(t_{k+1})-S(t_k)\right|^2 &= \sum_{k=0}^N \mathbf{E}\left|\int_{t_k}^{t_{k+1}}\int_0^\infty e^{-s}\tilde{x}^u(s)dse^{iu}du\right|^2 \\
&\leq \sum_{k=0}^N \mathbf{E}\int_{t_k}^{t_{k+1}} 1^2 du \int_{t_k}^{t_{k+1}}\left|\int_0^\infty e^{-s}\tilde{x}^u(s)dse^{iu}\right|^2 du \\
&\leq \delta\mathbf{E}\int_0^{t_{N+1}}\left\{\int_0^\infty 1^2 e^{-s}ds\cdot\int_0^\infty (\tilde{x}^u(s))^2 e^{-s}ds\right\}du \\
&= \delta\mathbf{E}\int_0^t \|\tilde{x}^u\|^2 du.
\end{aligned} \tag{2.31}$$

Inequality (2.31) holds for any natural number N . Thus by inequalities (2.17) and (2.31) it follows that

$$\lim_{\delta\to 0}\sum_{k=0}^N \mathbf{E}\left|S(t_{k+1})-S(t_k)\right|^2=0 \tag{2.32}$$

Observe also that the stochastic integral$T(t)$, $t\geq 0$ in (2.29)is a martingale satisfying $\mathbf{E}\left(T(t)\right)^2<\infty$ for all $t\geq 0$ and hence by equations (2.30) and (2.32) it follows that $Q(t)$ is a Dirichlet process as it satisfies Definition 1 of [4]. Let $\mathcal{D}$ denote the space of Dirichlet processes and define a subspace $\mathcal{D}_0\subseteq\mathcal{D}$ by

$$\mathcal{D}_0=\left\{S\in\mathcal{D}:\lim_{m\to\infty}\sup_n\sum_{k=m2^n}^\infty \mathbf{E}\left|S((k+1)2^{-n})-S(k2^n)\right|^2=0\right\}$$

Clearly the integrals $S(t)$ and $T(t)$ defined by (2.29) belong to $\mathcal{D}_0$, and so also $Q(t) \in \mathcal{D}_0$. Observe also that if λ is the Lebesgue measure and P our probability measure then for any $\epsilon > 0$ we have

$$\begin{aligned} &\lambda \otimes P \left\{ (u,\omega) : I_{[m,\infty)} \int_0^\infty e^{-s} \tilde{x}^u(s) ds > \epsilon \right\} \\ &\leq \frac{1}{\epsilon^2} \mathbf{E} \int_m^\infty \left(\int_0^\infty e^{-s} \tilde{x}^u(s) ds \right)^2 du \to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{2.33}$$

We also have

$$\frac{1}{2\pi} \mathbf{E} \int_{-\infty}^\infty \left| \int_0^\infty \int_0^\infty e^{-s} \tilde{x}^u(s) e^{i\xi u} ds\, du \right|^2 d\xi = \mathbf{E} \int_0^\infty \left(\int_0^\infty e^{-s} \tilde{x}^u(s) ds \right)^2 du \tag{2.34}$$

□

Comments:
Note that till now we have not finished studying $\lim_{t\to\infty} \|X_t\|^2$. But it is probable that the limiting situation is as follows:
The sum of the squares of the position and the velocity of the Dangling Spider namely $\big(x_1(t)^2 + x_2(t)^2\big)$ tends to a constant as $t \to \infty$. In other words the co-ordinates of the position and the velocity of the Dangling Spider form a circle for sufficiently large t. Let $t \mapsto \tilde{x}^t(\cdot)$ be the third component of $X_t \in X$, the solution to the delay S.D.E. (iv.1)(in [3]). (The space X is also called the history space.) We notice that $\lim_{t\to\infty} \|\tilde{x}^t(\cdot)\| = 0$: see assertion (b) of Theorem 1. Then the function $\xi \mapsto \int_0^\infty \int_0^\infty e^{-s} \tilde{x}^u(s) ds\, e^{i\xi u} du := L^2\text{-}\lim_{t\to\infty} \int_0^t \int_0^\infty e^{-s} \tilde{x}^u(s) ds\, e^{i\xi u} du$ can be interpreted as an L^2-function. More precisely, by Plancherel's theorem we have:

$$\begin{aligned} \frac{1}{2\pi} \mathbf{E} \int_{-\infty}^\infty \left| \int_0^\infty \int_0^\infty e^{-s} \tilde{x}^u(s) ds\, e^{i\xi u} du \right|^2 d\xi &= \mathbf{E} \int_0^\infty \left| \int_0^\infty e^{-s} \tilde{x}^u(s) ds \right|^2 du \\ &\leq \mathbf{E} \int_0^\infty \|\tilde{x}^u\|^2 du < \infty. \end{aligned}$$

2.2 Remarks:

(a) All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z : [0,a] \times \Omega \to \mathbf{R}$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities

$$\begin{aligned} |\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| &\leq K(t-s) \quad \text{and} \\ \mathbf{E}\left(|Z(t) - Z(s)|^2 |\mathcal{F}_s \right) &\leq K(t-s) \quad \text{for } 0 \leq s \leq t \leq a. \end{aligned}$$

Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work.See [2].

(b) All the results which we have established in this work, can be extended to a processes $f', g' : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to L(\mathbf{R}^m, \mathbf{R}^n)$ $(m, n \in \mathbf{N})$ instead of the processes $f, g : [0,a] \times \mathbf{R}^n \times L^2(J, \mathbf{R}^n) \to \mathbf{R}^n$ $(n \in \mathbf{N})$, and instead of the Brownian motion W we use the process $Z : [0,a] \times \Omega \to \mathbf{R}^m$ which is a martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$, continuous on $[0,a]$, and has independent increments and satisfies for some constant K the inequalities

$$|\mathbf{E}[Z(t) - Z(s)]|\mathcal{F}_s| \leq K(t-s) \quad \text{and} \quad \mathbf{E}\left(|Z(t) - Z(s)|^2 |\mathcal{F}_s \right) \leq K(t-s)$$

for $0 \leq s \leq t \leq a$. See [2].

(c) All the lemmas and theorems in this work hold for any delay interval $J' = [-r, 0) \quad (r \geq 0)$.See [2].

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